Solutions to short-answer questions

1 **a**

$$\begin{aligned} {}^{6}C_3 &= \frac{6!}{3!\cdot(6-3)!} \\ &= \frac{5!}{2!\cdot 3!} \\ &= \frac{6\cdot 5\cdot 4\cdot 3!}{3!\cdot 3!} \\ &= \frac{6\cdot 5\cdot 4\cdot 3!}{3!\cdot 3!} \\ &= \frac{6\cdot 5\cdot 4}{6} \\ &= 20 \end{aligned}$$

b

$$\begin{aligned} {}^{20}C_2 &= \frac{20!}{2!\cdot(20-2)!} \\ &= \frac{20!}{2!\cdot 18!} \\ &= \frac{20\cdot 19\cdot 18!}{2!\cdot 18!} \\ &= \frac{20\cdot 19}{2} \\ &= 190 \end{aligned}$$

c This can be evaluated by using the fact that ${}^{n}C_1 = n$. Otherwise, we simply use the formula for ${}^{n}C_r$ to give,

$$\begin{aligned} {}^{300}C_1 &= \frac{300!}{1!\cdot(300-1)!} \\ &= \frac{300!}{1!\cdot 299!} \\ &= \frac{300\cdot 299!}{299!} \\ &= 300. \end{aligned}$$

d

$$\begin{aligned} {}^{100}C_{98} &= \frac{100!}{98!\cdot(100-98)!} \\ &= \frac{100!}{98!\cdot 2!} \\ &= \frac{100\cdot 99\cdot 98!}{98!\cdot 2!} \\ &= \frac{100\cdot 99}{2} \\ &= 4950 \end{aligned}$$

2 We solve the following equation,

$$\begin{aligned} {}^{n}C_2 &= 55 \\ \frac{n!}{2!\cdot(n-2)!} &= 55 \\ \frac{n\cdot(n-1)\cdot(n-2)!}{2!\cdot(n-2)!} &= 55 \\ \frac{n\cdot(n-1)}{2} &= 55 \\ n(n-1) &= 110 \\ n^2-n-110 &= 0 \\ (n-11)(n+10) &= 0 \\ \Rightarrow n &= 11 \text{ as } n>0. \end{aligned}$$

3 **a** If the digits can be repeated there are 3 choices for each of the 3 positions. This gives a total of $3^3 = 27$ numbers.

b 3 digits can be arranged (without repetition) in $3! = 6$ ways.

4 There are 6 choices of student for the first position, 5 for the second and 4 for the third. This gives a total of $6 \times 5 \times 4 = 120$ arrangements.

5 There are 5 choices of desk for the first student, 4 for the second and 3 for the third. This gives a total of $5 \times 4 \times 3 = 60$ allocations.

6 There are ${}^4C_2$ ways of selecting 2 Year 11 students out of 4 and ${}^3C_2$ ways of selecting 2 Year 12 student out of 3. Using the Multiplication Principle gives a total of ${}^4C_2 \times {}^3C_2 = 6 \times 3 = 18$ selections.

7 Without restriction, there are ${}^7C_3 = 35$ ways of selecting 3 children out of 7. We then consider those selections that have no boy. There are are ${}^4C_3 = 4$ ways of selecting 3 of 4 girls. Therefore, there are $35 - 4 = 31$ selections that have at least one boy.

8 There are 5 flags, of which a group of 2 are alike, and a group of 3 are alike. These flags can be arranged in $\dfrac{5!}{2! \cdot 3!} = 10$ ways.

9 Label 26 holes with each of the letters from A to Z. Place each word in the hole according to its first letter. Since $53 = 2 \times 26 + 1$ there is some hole that contains at least 3 words.

10 Let N and B be the set of students playing netball and basketball respectively. Then using the Inclusion-Exclusion Principle gives,

$$\begin{aligned} |N \cup B| &= |N| + |B| - |N \cap B| \\ 20 &= |N| + 12 - 4 \\ 20 &= |N| + 8 \\ |N| &= 12. \end{aligned}$$

So 12 students play netball.

11 Let's denote the six people by letters A, B, C, D, E and F. First suppose person C is between person A and B. We then have to arrange four items: $\{A, C, B\}, D, E, F$. This can be done in $4!$ ways. Person C is fixed but A and B can be arranged in 2 ways. We then multiply by 4 as there are 4 different people who can go between A and B. This gives a total of $4! \times 2 \times 4 = 192$ arrangements.

Solutions to multiple-choice questions

1 **C** Using the Multiplication Principle there are $4 \times 3 \times 4 = 48$ ways that Sam can select his remaining three subjects.

2 **B** Using both the Addition and Multiplication Principle gives a total of $3 + 2 \times 4 = 11$ different paths.

3 **A** 10 people can be arranged in $10!$ ways.

4 **D** The first digit can be chosen in 6 ways, the second digit in 5 ways on the third digit in 4 ways. This gives a total of $6 \times 4 \times 3$ arrangements.

5 **B** There are 4 vowels, so there are 4 choices for the first letter and 3 for the last. After filling these two positions, there are 4 choices for the second letter, 3 for the third, and so on. This gives a total of $4 \times 4 \times 3 \times 2 \times 1 \times 3 = 288$ arrangements.

6 **B** There are 7 flags in total, of which a group of 4 are alike and another group of 3 are alike. These can be arranged in $\dfrac{7!}{4! \cdot 3!}$ ways.

7 **C** 3 items can be chosen out of 9 in ${}^9C_3$ ways.

8 **D** A set with 6 elements has 2^6 subsets. One of these is the empty set, so there are $2^6 - 1$ subsets with at least one element.

9 **C** There are ${}^9C_2$ ways of selecting 2 girls out of 9 and ${}^8C_2$ ways of selecting 2 boys out of 8. Using the Multiplication Principle gives a total of ${}^9C_2 \times {}^8C_2$ selections.

10 **C** Label two holes with the colours blue and red. Select 5 balls. Then since $5 = 2 \times 2 + 1$, there is some hole that contains at least 3 balls. Clearly 5 is the smallest number since selecting 4 balls might give 2 blue and 2 red balls.

11 **A** Let F, C and G be the sets of students studying French, Chinese and German, respectively. Then using the Exclusion-Inclusion Principle gives,

$$\begin{aligned}|F \cup C \cup G| =& |F| + |C| + |G| \\ & -|F \cap C| - |F \cap G| - |C \cap G| \\ & +|F \cap C \cap G| \\ 30 =& 15 + 15 + 17 \\ & -|F \cap C| - |F \cap G| - |C \cap G| \\ & +|F \cap C \cap G|\end{aligned}$$

Therefore,

$$17 = |F \cap C| + |F \cap G| + |C \cap G| - |F \cap C \cap G|$$

Since 15 students study more than one subject we also know that,

$$15 = |F \cap C| + F \cap G| + |C \cap G| - 2|F \cap C \cap G|.$$

Subtracting the second equation from the first gives,

$$|F \cap C \cap G| = 2.$$

Solutions to extended-response questions

1 **a** The first digit is 5 so there is only 1 choice for that position. The remaining 5 numbers can be arranged in $5!$ ways. This gives a total of $1 \times 5! = 120$ arrangements.

b If the first digit is even then there are 3 choices for that position. The remaining 5 numbers can be arranged in $5!$ ways. This gives a total of $3 \times 5! = 360$ arrangements.

c There are 2 choices for if the first number is even or odd. The 3 odd numbers can then be arranged in $3!$ ways, and the 3 even numbers can be arranged in $3!$ ways. This gives a total of $2 \times 3! \times 3! = 72$ arrangements.

d If we group the even numbers, we need only arrange 4 items: $1, 3, 5, \{2, 4, 6\}$ and $\{2, 4, 6\}$. This can be done in $4!$ ways. The 3 even numbers can then be arranged in $3!$ ways. This gives a total of $4! \times 3! = 144$ arrangements.

2 **a** If the first letter is E, then there is only 1 choice for that position. There are 5 remaining choices for the second letter, and 4 for the third. This gives a total of $1 \times 5 \times 4 = 20$ arrangements.

b If the first letter is a vowel then there are 4 choices for that position. There are 5 remaining choices for the second letter and 4 for the third. This gives a total of $4 \times 5 \times 4 = 80$ arrangements.

c If the letter E is used then there are 3 choices of position for that letter. There are 5 and then 4 choices for the remaining two positions. This gives a total of $3 \times 5 \times 4 = 60$ arrangements.

3 **a** 4 students can be chosen from 10 in ${}^{10}C_4 = 210$ ways.

b If the school captain must be chosen then we are still to chose 3 students from the 9 that remain. This can be done in ${}^9C_3 = 84$ ways.

c We must select 2 boys from 4 and 2 girls from 6. This can be done in ${}^4C_2 \times {}^6C_2 = 6 \times 15 = 90$ ways.

d The total number of selections is ${}^{10}C_4 = 210$. The number of selections with no boys is ${}^{6}C_4 = 15$. Therefore, the number of selections with at least one boy will be $210 - 15 = 195$.

4 a There are 8 letters in total, 2 of which are N, 2 of which are J and 4 of which are T. These can be arranged in $\dfrac{8!}{2! \cdot 2! \cdot 4!} = 420$ ways.

b The first and last letter are both N. Therefore, we need only arrange 6 letters, 2 of which are J and 4 of which are T. This can be done in $\dfrac{6!}{2! \cdot 4!} = 15$ ways.

c We group the two J's so that we now must arrange {J,J},N,N,T,T,T,T. There are 7 items to arrange, of which 2 are N and 4 are T. This can be done in $\dfrac{7!}{2! \cdot 4!} = 105$ ways.

d If no two T's are adjacent then they are either all in the even positions or all in the odd positions.
The 4 remaining letters are: N,N,J,J. These can be arranged in $\dfrac{4!}{2! \cdot 2!} = 6$ ways. We multiply this by 2 because the J's can be placed in either even or odd positions. This gives a total of $6 \times 2 = 12$ arrangements.

5 a i 3 toppings can be chosen from 6 in ${}^{6}C_3 = 20$ ways.

ii 2 additional toppings have to be chosen from among 5 that remain. Thiscan be done in ${}^{5}C_2 = 10$ ways.

iii A set of size 6 has $2^6 = 64$ subsets.

b We want to find the smallest value of n such that $2^n > 200$. Since $2^7 = 128$ and $2^8 = 256$ they must use at least 8 different toppings.

6 a 4 people can be chosen from 10 in ${}^{10}C_4 = 210$ ways.

b We must choose 2 women from 5 and 2 men from 5. Using the Multiplication Principle, this can be done in ${}^{5}C_2 \times 5C_2 = 100$ ways.

c We must choose 2 married couples from 5. This can be done in ${}^{5}C_2 = 10$ ways.

d We first chose 4 couples from 5. This can be done in ${}^{5}C_4 = 5$ ways. Then from each of the 4 couples there are 2 choices of either husband or wife. Using the Multiplication Principles gives a total of $5 \times 2 \times 2 \times 2 \times 2 = 80$ selections.

7 a There are 26 choices for both the first and second letter so there are $26 \times 26 = 676$ two letter initials.

b We first consider the two letter initials that contain no vowel. There are 21 choices of consonant for each of the two letters, giving a total of $21 \times 21 = 441$. Therefore, $676 - 441 = 235$ two letter initials contain at least one vowel.

c Label 676 holes with each of the different two letter initials. Since $50000/676 > 73$, there must be some hole containing 74 items. Therefore, there are at least 74 people who share the same initials.

8 a The lowest common multiple is 24.

b $A \cap B$ consists of numbers that are multiples of 6 and 8. That is, it contains multiples of 24. Therefore, $A \cap B = \{24, 48, 72, 96\}$, so there are 4 elements.

c We have,

A	multiples of 6	$\lvert A\rvert = 96 \div 6 = 16$
B	multiples of 8	$\lvert B\rvert = 96 \div 8 = 12$
$A \cap B$	multiples of 24	$\lvert A \cap B\rvert = 96 \div 24 = 4$

Using the Inclusion-Exclusion Principle gives,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ &= 16 + 12 - 4 \\ &= 24.\end{aligned}$$

d There are 24 numbers that are divisible by 6 or 8 and $96 - 24 = 72$ that are not. Therefore, the probability that the integer is not divisible by 6 or 8 is $\frac{72}{96} = \frac{3}{4}$.

9 a Let N be a movement of one unit in the north direction, and E be a movement of one unit in the east direction. Then each path from H to G is described by six N's and six E's in some order. As there are twelve letters in total, there are $\frac{12!}{6! \cdot 6!} = 924$ paths.

b In three years there are at least $365 \times 3 = 1095$ days and there are 924 different paths. Since $1095 > 924$, by the Pigeonhole Principle, there is some path taken at least twice in the course of 3 years.

c i Each path from H to C is described by two N's and two E's in someorder. As there are four letters in total, there are $\frac{4!}{2! \cdot 2!} = 6$ paths.

ii Each path from C to G is described by four N's and four E's in someorder. As there are eight letters in total, there are $\frac{8!}{4! \cdot 4!} = 70$ paths.

iii There are 6 paths from H to C and 70 paths from C to G. Using theMultiplication Principle gives a total of $6 \times 70 = 420$ paths from H to C to G.

d Let X be the set of paths from H to C to G. We have found that $|X| = 420$. Let Y be the set of paths from H to B to G. Clearly, $|Y| = 420$.
Now $X \cap Y$ is the set of paths from H to C to B to G. Since there are 6 paths from each of three steps, $|X \cap Y| = 6 \times 6 \times 6 = 216$. Lastly, applying the Inclusion-Exclusion Principle gives,

$$\begin{aligned}|X \cup Y| &= |X| + |Y| - |X \cap Y| \\ &= 420 + 420 - 216 \\ &= 624\end{aligned}$$

10 The ratio of green to yellow to orange balls is $1 : 3 : 6 = 2 : 6 : 12$. Therefore, the ratio of blue to red to green to yellow to orange balls is $1 : 4 : 2 : 6 : 12$. Using this, we can find the number of balls of each type. This gives:

blue $\frac{1}{25} \times 400 = 16$

red $\frac{4}{25} \times 400 = 64$

green $\frac{2}{25} \times 400 = 32$

yellow $\frac{6}{25} \times 400 = 96$

orange $\frac{12}{25} \times 400 = 192$

Label five holes with each of the above colours. In the worst case, you might first select 16 blue, 32 green, 49 red, 49 yellow and 49 orange. Selecting one additional ball must give either a red, yellow or orange ball. This gives a total of $16 + 32 + 3 \times 49 + 1 = 196$ balls.